Simple Derivation of the Hermite Bicubic Patch using Tensor Product

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Abstract Bicubic parametric patches are widely used in various geometric applications. These patches are critical in CAD/CAM systems, which are applied in the automotive industry and mechanical and civil engineering. Commonly, Hermite, Bézier, Coons, or NURBS patches are employed in practice. However, the construction of the Hermite bicubic patch is often not easy to explain formally. This contribution presents a new formal method for constructing the Hermite bicubic plate based on the tensor product approach.

Keywords: tensor product, Hermite curve, Hermite patch, interpolation, parametric patches, Kronecker product.

1. Introduction

This contribution introduces a simplified approach to deriving the Hermite bicubic parametric patch. A clear and formal derivation is essential for a proper understanding, particularly in computer graphics and geometric modeling courses, where only the final mathematical form is typically shown. The standard derivation of the Hermite form is often considered complex.

The method presented here is based on the tensor product with linear operators. It is straightforward, easy to follow, and well-suited for introductory courses. The Bézier parametric patch $S(u,v)$, as defined by Bézier [2], is based on the tensor product of Bézier curves:

$$S(u,v) = C(u) \otimes C(v)\,.$$

In general, cubic parametric curves and bicubic patches are discussed in works such as Cogen [3], Goldman [5], Prautzsch [9], Holliday [6], and Rockwood [10].

It should be noted that while the boundary curves of a bicubic patch are cubic, the diagonal and anti-diagonal curves (for $u = v$ and $u = 1 - v$) are of degree 6. If the degree is limited to 3, additional conditions must be applied. Such constraints were formulated for Hermite patches in Skala [14] and Bézier patches in Kolcun [7] and Skala [13]. A geometric interpretation of the diagonal in a Bézier volume was explored by Holliday and Farin [6]. Triangular patches are discussed in Farin [4].

2. Tensor product

The tensor product [17] is not commonly used in basic courses, yet it is a powerful and versatile tool. It defines a non-commutative product of two vectors, $\mathbf{v} = [v_1, v_2, \dots, v_n]^T$ and $\mathbf{w} = [w_1, w_2, \dots, w_m]^T$, as:

$$\mathbf{v} \otimes \mathbf{w} = \begin{bmatrix} v_1 w_1 & v_1 w_2 & \cdots & v_1 w_m \\ v_2 w_1 & v_2 w_2 & \cdots & v_2 w_m \\ \vdots & \vdots & \ddots & \vdots \\ v_n w_1 & v_n w_2 & \dots & v_n w_m \end{bmatrix} \tag{1}$$

The Kronecker product [15], named after Leopold Kronecker, generalizes the outer product and is a specific case of the tensor product. The Kronecker product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined as:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \otimes \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} =$$

$$\begin{bmatrix} a_{1,1} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{1,2} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \\ a_{2,1} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} & a_{2,2} \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix} . \tag{2}$$

Applying the tensor product to differential operators yields [12]:

$$\begin{bmatrix} 1 \\ \frac{\partial}{\partial u} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \frac{\partial}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\partial}{\partial v} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial u}(\frac{\partial}{\partial v}) \end{bmatrix} = \begin{bmatrix} 1 & \frac{\partial}{\partial v} \\ \frac{\partial}{\partial u} & \frac{\partial^2}{\partial u \partial v} \end{bmatrix} . \tag{3}$$

Both the tensor and the Kronecker products are multilinear [16] and can also be applied to functions [8].

3. Hermite curve using tensor product

The Hermite parametric cubic curve segment uses two end-points x_1, x_2 and two tangential vectors x_3, x_4 of the cubic segment end-points, see Fig. 1.

The position of the point $x(u)$ is given by (4):

$$x(u) = a_1 u^3 + a_2 u^2 + a_3 u + a_4 \, , \quad x(u) = \sum_{i=1}^{4} a_i u^{4-i} \, , \quad u \in < 0, 1 > , \tag{4}$$

and the tangent vector $x'(u) = \frac{dx(u)}{du}$ is given by (5):

$$x^{(u)}(u) = 3a_1 u^2 + 2a_2 u + a_3 \, , \quad x^{(u)}(u) = \sum_{i=1}^{3} (4-i) \; a_i u^{3-i} \, , \quad u \in < 0, 1 > \; . \tag{5}$$

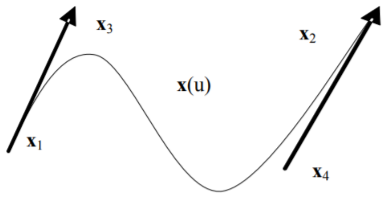

Fig. 1. Hermite cubic curve (tangential vectors shortened); source: own.

The Eq. (4) can be rewritten using the *dot product* as:

$$x(u) = [a_1, a_2, a_3, a_4]^T \ [u^3, u^2, u, 1] = \mathbf{a}^T \mathbf{u} \,. \tag{6}$$

Solving (4) and (5) for the curve segment end-points, i.e. $u = 0$ and $u = 1$, the following system of linear equations is obtained:

$$x(0) = a_4 \,, \quad x(1) = a_1 + a_2 + a_3 + a_4 \,, \quad x^{(u)}(0) = a_3 \,, \quad x^{(u)}(1) = 3a_1 + 2a_2 + a_3 \,, \tag{7}$$

where $x^{(u)} = \frac{\partial x}{\partial u}$.

This leads to a system of equations for the unknown coefficients $\mathbf{a} = [a_1, a_2, a_3, a_4]^T$ for the given end-points property $\boldsymbol{\xi} = [x(0), x(1), x^{(u)}(0), x^{(u)}(1)]^T \stackrel{\text{def}}{=} [x_1, x_2, x_1^{(u)}, x_2^{(u)}]^T$.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1^{(u)} \\ x_2^{(u)} \end{bmatrix} , \quad \mathbf{Ba} = \boldsymbol{\xi} \,. \tag{8}$$

Solving the linear system of equations $\mathbf{Ba} = \boldsymbol{\xi}$, Eq. (7), the coefficients of the Hermite form are obtained. Then

$$x(u) = a_1 u^3 + a_2 u^2 + a_3 u + a_4 = (\mathbf{B}^{-1}\boldsymbol{\xi})^T \mathbf{u} = \boldsymbol{\xi}^T \mathbf{B}^{-T} \mathbf{u} \,, \tag{9}$$

where $\mathbf{u} = [u^3, u^2, u, 1]^T$, $\mathbf{B}^{-T}$ is the transposed inverse matrix.

Now, the Hermite parametric curve is then described as:

$$x(u) = \boldsymbol{\xi}^T \mathbf{M}_H \mathbf{u} = \boldsymbol{\xi}^T \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \mathbf{u} = \mathbf{u}^T \mathbf{M}_H^T \boldsymbol{\xi} \,, \tag{10}$$

where $\mathbf{u} = [u^3, u^2, u, 1]^T$, $\mathbf{M}_H = \mathbf{B}^{-1}$ is the matrix of the Hermite form, and $\boldsymbol{\xi} = [x(0), x(1), x^{(u)}(0), x^{(u)}(1)]^T \equiv [x_1, x_2, x_1^{(u)}, x_2^{(u)}]^T$ are the control values of the curve $x(u)$.

It should be noted that the Equation (10) represents only the $x(u)$-coordinate and for the other coordinates, i.e. $y(u)$, $z(u)$, it is similar.

Generally, for the E^3 case, for a curve $\mathbf{C}(u) = [x(u), y(u), z(u)]^T$ we can write:

$$\mathbf{C}(u) = [\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4]^T\ \mathbf{M}_H\ [u^3, u^2, u, 1] = [\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4]^T\ \mathbf{M}_H\ \mathbf{u}\,, \tag{11}$$

where $\mathbf{P}_1 = [x_1, y_1, z_1]^T$, $\mathbf{P}_2 = [x_2, y_2, z_2]^T$ are vectors of the curve end-points, $\mathbf{P}_3 = [x_1^{(u)}, y_1^{(u)}, z_1^{(u)}]^T$, $\mathbf{P}_4 = [x_2^{(u)}, y_2^{(u)}, z_2^{(u)}]^T$ are vectors of the tangential vectors at the curve end-points and $\mathbf{u} = [u^3, u^2, u, 1]^T$.

The Equation (10) can be rewritten as:

$$\mathbf{C}(u) = [\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4]^T\ \mathbf{M}_H\ \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}\,, \tag{12}$$

i.e.

$$\begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix} = \begin{bmatrix} x_1^{(u)} & x_2^{(u)} & x_3^{(u)} & x_4^{(u)} \\ y_1^{(u)} & y_2^{(u)} & y_3^{(u)} & y_4^{(u)} \\ z_1^{(u)} & z_2^{(u)} & z_3^{(u)} & z_4^{(u)} \end{bmatrix}\ \mathbf{M}_H\ \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}\,. \tag{13}$$

Note that the Eq. (10) is formally valid also for the Bézier, Catmul, Ferguson, etc. curves; however, the control vector $\boldsymbol{\xi}$ has different properties.

The Bézier curve of the degree n is defined as:

$${}^{(B)}x(u) = \sum_{i=0}^{n} x_i \binom{n}{i} u^i (1-u)^{n-i}\,, \tag{14}$$

and the tangential vectors are defined at the end-points as:

$$x^{(u)}(0) = n(x_1 - x_0) \quad x^{(u)}(1) = n(x_n - x_{n-1})\,. \tag{15}$$

There is a direct connection between the Hermite and Bézier forms. Therefore, the Hermite, Bézier, Ferguson, etc., curves are mutually convertible, see Anand [1].

It should be noted that an invertible matrix $\mathbf{M}_{H\to B}$ (4×4) exists, which transforms the Hermite form to the Bézier form:

$${}^{(B)}x(u) = \mathbf{M}_{H\to B}\ {}^{(H)}x(u)\,, \tag{16}$$

where ${}^{(B)}x(u)$, resp. ${}^{(H)}x(u)$, means the x-coordinate of the Hermite cubic curve, resp. Bézier cubic curve, see Anand [1]. Continuity conditions were also studied in Skala [11].

4. Hermite patch using tensor product

The two-dimensional case of the multi-variate Hermite interpolation is the Hermite bicubic patch. The bicubic parametric patch for the x-coordinate is defined as:

$$x(u,v) = \left(\sum_{i=1}^{4} a_i u^{4-i}\right)\left(\sum_{j=1}^{4} b_j v^{4-j}\right) = \sum_{i=1}^{4}\sum_{j=1}^{4} a_i b_j \; u^{4-i} v^{4-j}\,, \tag{17}$$

$$x(u,v) = \mathbf{u}^T \mathbf{S}\;\mathbf{v}\,, \quad s_{i,j} = a_i b_j\,, \tag{18}$$

where: $\mathbf{u} = [u^3, u^2, u, 1]^T$, $\mathbf{v} = [v^3, v^2, v, 1]^T$ and the matrix $\mathbf{S}$ (4×4) has the $s_{i,j}$ elements. Similarly, for the $y(u,v)$ and $z(u,v)$ coordinates.

Using the tensor product on functions, a simple formula is obtained [12]:

$$x(u,v) = x(u) \otimes x(v)\,. \tag{19}$$

The Eq. (18) describes a parametric patch $x(u,v)$. Each point of the curve $x(u)$ in (10) is parameterized by the second parameter v as:

$$x(u,v) = \mathbf{u}^T \mathbf{M}_H^T \boldsymbol{\xi}(\mathbf{v})\,, \tag{20}$$

where: $\boldsymbol{\xi}(\mathbf{v}) = [x_1(v), x_2(v), x_1^{(v)}(u), x_2^{(v)}(u)]^T$. It should be noted that all elements of the vector $\boldsymbol{\xi}(\mathbf{v})$ are the Hermite curves again. It means, that

$$\begin{aligned} x_1(v) &= [x_{11}, x_{12}, x_{11}^{(v)}, x_{12}^{(v)}]\mathbf{M}_H \mathbf{v}\,, \\ x_2(v) &= [x_{21}, x_{22}, x_{21}^{(v)}, x_{22}^{(v)}]\mathbf{M}_H \mathbf{v}\,, \\ x_1^{(v)}(u) &= [x_{11}^{(u)}, x_{12}^{(u)}, x_{11}^{(uv)}, x_{12}^{(uv)}]\mathbf{M}_H \mathbf{v}\,, \\ x_2^{(v)}(u) &= [x_{21}^{(u)}, x_{22}^{(u)}, x_{21}^{(uv)}, x_{22}^{(uv)}]\mathbf{M}_H \mathbf{v}\,, \end{aligned} \tag{21}$$

where $x^{(uv)} \stackrel{\text{def}}{=} \frac{\partial^2 x}{\partial u \partial v}$ and $\mathbf{v} = [v^3, v^2, v, 1]^T$.

Now, the Hermite patch $x(u,v)$ for the x-coordinate can be rewritten as:

$$x(u,v) = \mathbf{u}^T \mathbf{M}_H^T \begin{bmatrix} x_{11} & x_{12} & x_{11}^{(v)} & x_{12}^{(v)} \\ x_{21} & x_{22} & x_{21}^{(v)} & x_{22}^{(v)} \\ x_{11}^{(u)} & x_{12}^{(u)} & x_{11}^{(uv)} & x_{12}^{(uv)} \\ x_{21}^{(u)} & x_{22}^{(u)} & x_{21}^{(uv)} & x_{22}^{(uv)} \end{bmatrix} \mathbf{M}_H \mathbf{v}\,, \tag{22}$$

or using more compact form:

$$x(u,v) = \mathbf{u}^T \mathbf{M}_H^T \;\mathbf{X}\; \mathbf{M}_H \mathbf{v}\,, \tag{23}$$

where the matrix $\mathbf{X}$ is the matrix of the control values of the Hermite patch form. It can be seen, that it is the biquadratic form. This formal notation is common for the other bicubic patches, e.g., for the Bézier, Ferguson, etc.

5. Hermite bicubic plate using tensor product

The Hermite bicubic plate can be expressed by using tensor product as:

$$\mathbf{S}(u,v) = \mathbf{C}(u) \otimes \mathbf{C}(v)\,. \tag{24}$$

Using the tensor product and more compact form with the block matrix notation:

$$x(u,v) = \mathbf{u}^T \mathbf{M}_H^T \begin{bmatrix} x_{11} & x_{12} & x_{11}^{(v)} & x_{12}^{(v)} \\ x_{21} & x_{22} & x_{21}^{(v)} & x_{22}^{(v)} \\ x_{11}^{(u)} & x_{12}^{(u)} & x_{11}^{(uv)} & x_{12}^{(uv)} \\ x_{21}^{(u)} & x_{22}^{(u)} & x_{21}^{(uv)} & x_{22}^{(uv)} \end{bmatrix} \mathbf{M}_H \mathbf{v}\,. \tag{25}$$

The Eq. (25) can be rewritten to:

$$x(u,v) = \mathbf{u}^T \mathbf{M}_H^T \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \mathbf{M}_H \mathbf{v}\,, \tag{26}$$

where x_{ij} are the control values, see Fig. 2.

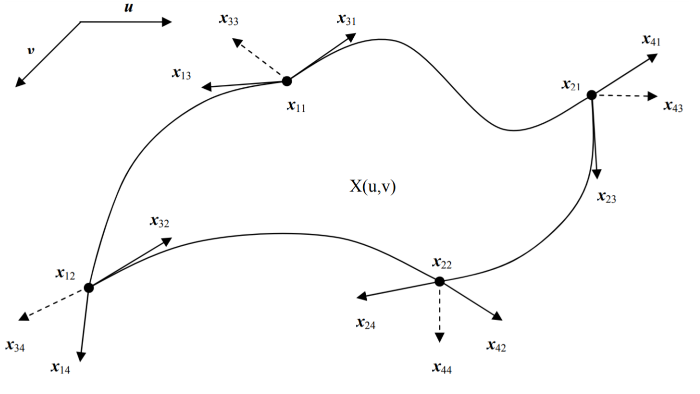

Fig. 2. Hermite bi-cubic patch (tangential and twist vectors scaled); source: own.

Now, using the tensor product and a submatrix 4×4 of the patch end-points a more "compact form" describing the Hermite patch is obtained as:

$$\mathbf{P}(u,v) = \mathbf{u}^T \mathbf{M}_H^T \begin{bmatrix} 1 & \frac{\partial}{\partial v} \\ \frac{\partial}{\partial u} & \frac{\partial^2}{\partial u \partial v} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \mathbf{M}_H \mathbf{v} =$$

$$\mathbf{u}^T \mathbf{M}_H^T \begin{bmatrix} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} & \frac{\partial}{\partial v} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \\ \frac{\partial}{\partial u} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} & \frac{\partial^2}{\partial u \partial v} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \end{bmatrix} \mathbf{M}_H \mathbf{v} \,. \tag{27}$$

If the differential tensor operator (3) is applied on the Hermit bicubic corners, the matrix form is obtained as follows:

$$\begin{bmatrix} 1 & \frac{\partial}{\partial v} \\ \frac{\partial}{\partial u} & \frac{\partial^2}{\partial u \partial v} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} & \frac{\partial}{\partial v} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \\ \frac{\partial}{\partial u} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} & \frac{\partial^2}{\partial u \partial v} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \frac{\partial}{\partial u}\mathbf{P}_{11} & \frac{\partial}{\partial u}\mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} & \frac{\partial}{\partial u}\mathbf{P}_{21} & \frac{\partial}{\partial u}\mathbf{P}_{22} \\ \frac{\partial}{\partial v}\mathbf{P}_{11} & \frac{\partial}{\partial v}\mathbf{P}_{12} & \frac{\partial^2}{\partial u \partial v}\mathbf{P}_{11} & \frac{\partial^2}{\partial u \partial v}\mathbf{P}_{12} \\ \frac{\partial}{\partial v}\mathbf{P}_{21} & \frac{\partial}{\partial v}\mathbf{P}_{22} & \frac{\partial^2}{\partial u \partial v}\mathbf{P}_{21} & \frac{\partial^2}{\partial u \partial v}\mathbf{P}_{22} \end{bmatrix} . \tag{28}$$

It can be seen that this matrix clearly shows the Hermite form properties.

It should be noted that the Hermite bicubic parametric patch can be converted to the Bézier bicubic patch similarly to the case of cubic curves, see Anand [1].

Using a more formally general form:

$$\begin{aligned} \mathbf{S}(u,v) &= \left(\mathbf{u}^T \mathbf{M}_H^T \begin{bmatrix} 1 \\ \frac{\partial}{\partial u} \end{bmatrix} \right) \otimes \left(\begin{bmatrix} 1 & \frac{\partial}{\partial v} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \right) \mathbf{M}_H \mathbf{v} \\ &= \mathbf{u}^T \mathbf{M}_H^T \left(\begin{bmatrix} 1 & \frac{\partial}{\partial v} \\ \frac{\partial}{\partial u} & \frac{\partial^2}{\partial u \partial v} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \right) \mathbf{M}_H \mathbf{v} \,. \end{aligned} \tag{29}$$

Then using algebraic manipulations, the final formula is obtained:

$$\mathbf{S}(u,v) = \mathbf{u}^T \mathbf{M}_H^T \begin{bmatrix} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} & \frac{\partial}{\partial v} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \\ \frac{\partial}{\partial u} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} & \frac{\partial^2}{\partial u \partial v} \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \end{bmatrix} \mathbf{M}_H \mathbf{v} \,. \tag{30}$$

It should be noted that Eq. (30) can be rewritten to a more general form valid for the Bézier, Catmull-Rom, etc. patches, as:

$$x(u,v) = \mathbf{u}^T \mathbf{M}_F^T\ \mathbf{X}\ \mathbf{M}_F \mathbf{v}\,,\ y(u,v) = \mathbf{u}^T \mathbf{M}_F^T\ \mathbf{Y}\ \mathbf{M}_F \mathbf{v}\,,\ z(u,v) = \mathbf{u}^T \mathbf{M}_F^T\ \mathbf{Z}\ \mathbf{M}_F \mathbf{v}\,, \tag{31}$$

where $\mathbf{M}_F$ is a matrix of the form, i.e. Hermite, Bézier, Catmull-Rom, etc., and $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are the matrices of the control values of the form used.

In the Hermite patch case, parameterization of the x-coordinate is given as:

$$x(u,v) = \mathbf{u}^T \mathbf{M}_H^T \begin{bmatrix} x_{11} & x_{12} & x_{11}^{(v)} & x_{12}^{(v)} \\ x_{21} & x_{22} & x_{21}^{(v)} & x_{22}^{(v)} \\ x_{11}^{(u)} & x_{12}^{(u)} & x_{11}^{(uv)} & x_{12}^{(uv)} \\ x_{21}^{(u)} & x_{22}^{(u)} & x_{21}^{(uv)} & x_{22}^{(uv)} \end{bmatrix} \mathbf{M}_H \mathbf{v}\,; \tag{32}$$

then (32) can be formally simplified further:

$$x(u,v) = \mathbf{u}^T \mathbf{M}_H^T \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \mathbf{M}_H \mathbf{v}\,, \tag{33}$$

where x_{ij} represent the control values, as shown in Fig. 2, and similarly for $y(u,v)$ and $z(u,v)$ coordinates.

This matrix clearly illustrates the properties of the Hermite form.

6. Conclusion

This contribution presents an alternative to the Hermite cubic curve and Hermite bicubic patch derivation using the tensor product. The tensor product can be applied not only to vectors and matrices but also to functions.

In many computer graphics courses, only the vector or matrix form is shown, without a detailed derivation of the formulas or a detailed, "boring and lengthy" derivation.

Using the tensor matrix operations makes the derivation of the Hermite form clearer, especially for the Hermite bicubic patch.

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